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# On the Order Optimality of Large-scale Underwater Networks—Part II: Dense Network Model

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**Abstract** This is the second in a two-part series of papers on information-theoretic capacity scaling laws for an underwater acoustic network with $n$ regularly located nodes on a square, in which both bandwidth and received signal power can be limited significantly. Part I showed that in extended networks of unit node density, using the nearest-neighbor multi-hop (MH) protocol is order-optimal for all operating regimes. Part II shows the analysis for a dense network of unit area. As in Part I, we assume a narrow-band model where the carrier frequency scales as a function of $n$, and then characterize an attenuation parameter depending on the frequency scaling as well as the transmission distance. In dense networks, by deriving a cut-set upper bound on the throughput scaling, we show that there exists either a bandwidth or power limitation, or both, according to the path-loss attenuation regimes, thus yielding the upper bound that has three fundamentally different operating regimes. As similarly in Part I, an achievability result based on the MH transmission, which is suitable due to the low propagation speed of acoustic channel, is also presented in dense networks. The operating regimes that guarantee the order optimality

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are then identified. It thus turns out that frequency scaling is instrumental towards achieving the order optimality in the regimes. As a result, vital information for the fundamental limits of dense underwater networks is provided by showing capacity scaling laws.

**Keywords** Achievability · Attenuation parameter · Bandwidth · Capacity scaling law · Carrier frequency · Cut-set upper bound · Dense network · Multi-hop (MH) · Operating regime · Power · Underwater acoustic network

## 1 Introduction

In Part I [1] of this two-part series, we studied a large-scale underwater acoustic network of unit node density (i.e., an extended network [2–6]), in which both bandwidth and received signal power can be limited significantly, and analyzed its total capacity scaling, first introduced by Gupta and Kumar [7] in wireless radio networks. More specifically, we characterized the connection between the number of nodes, $n$, the carrier frequency, and the sum throughput while allowing the frequency scaling with $n$. Both an information-theoretic upper bound and an achievable rate scaling were derived in extended regular networks, and then using the nearest-neighbor multi-hop (MH) routing strategy was shown to be order-optimal for all operating regimes.

This paper, Part II in the two-part series, deals with an underwater network of unit area, i.e., a *dense* network [4, 7, 8], considered as another fundamental network model together with an extended network shown in Part I, for case where the carrier frequency scales as a certain function of $n$ in a narrow-band model. We thus study both upper and lower bounds on the total throughput scaling in a dense underwater network.

We explicitly characterize an attenuation parameter depending on the frequency scaling as well as the transmission distance, and then identify fundamental path-loss attenuation regimes according to the parameter. For dense networks with $n$ regularly distributed nodes on a square, we derive an upper bound on the total throughput scaling using the cut-set bound. Our upper bound basically follows the arguments, similarly as in [9]: there exists either a bandwidth or power limitation, or both, according to the operating regimes. Our results hence indicate that the upper bound for dense networks has three fundamentally different operating regimes according to the attenuation parameter. Specifically, the network is bandwidth-limited as the path-loss attenuation is small. This is because network performance on the total throughput is roughly linear in the bandwidth. However, at the medium attenuation regime, the network is both bandwidth- and power-limited since the amount of available bandwidth and received signal power affects the performance. Finally, the network becomes power-limited as the attenuation parameter increases exponentially with respect to more than $\sqrt{n}$, i.e., as the frequency scales faster than or as $n^{1/4}$, which corresponds to the high attenuation regime. As similarly in Part I, we also present a constructive achievability result based on the nearest-neighbor MH transmission, which is suitable for underwater networks due to the very long propagation delay of acoustic signal in water [17], in dense regular networks. We identify the operating regimes such that the optimal capacity scaling is guaranteed. Therefore, this key result indicates that frequency scaling is instrumental towards achieving the order optimality in the regimes.

The rest of this paper is organized as follows. Section 2 describes our system and channel models. In Section 3, the cut-set upper bound on the throughput is derived. In Section 4, the achievable throughput scaling is analyzed. Finally, Section 5 summarizes the paper with some concluding remarks.

Throughout this paper, $[\cdot]_{ki}$ denotes the $(k,i)$-th element of a matrix. $\mathbf{I}_n$ is the identity matrix of size $n \times n$, $\det(\cdot)$ is the determinant, and $|\mathcal{X}|$ is the cardinality of the set $\mathcal{X}$. $\mathbb{C}$ is the field of complex numbers and $E[\cdot]$ is the expectation. Unless otherwise stated, all logarithms are assumed to be to the base 2.

## 2 System and Channel Models

In order to set up the discussions in this paper, we reiterate the system and channel models described in Part I. Our model used in this paper follows exactly the same line [1] as the extended network case except for per-node distance.

We consider a two-dimensional underwater network that consists of $n$ nodes on a square such that two neighboring nodes are $1/\sqrt{n}$ unit of distance apart from each other, i.e., a regular network [5,6]. We randomly pick a matching of S–D pairs. Each node has an average transmit power constraint $P$ (constant), and we assume that the channel state information (CSI) is available at all receivers, but not at the transmitters. It is assumed that each node transmits at a rate $T(n)/n$, where $T(n)$ denotes the total throughput of the network.

Now let us turn to channel modeling. We assume frequency-flat channel of bandwidth $W$ Hz around carrier frequency $f$, which satisfies $f \gg W$, i.e., narrow-band model. We simply focus on a line-of-sight channel between each pair of nodes. An underwater acoustic channel is characterized by an attenuation that depends on both the distance $r_{ki}$ between nodes $i$ and $k$ $(i, k \in \{1, \cdots, n\})$ and the signal frequency $f$, and is given by

$$A(r_{ki}, f) = c_0 r_{ki}^{\alpha} a(f)^{r_{ki}} \tag{1}$$

for some constant $c_0 > 0$ independent of $n$, where $1 \le \alpha \le 2$ is the spreading factor and $a(f) > 1$ is the absorption coefficient [11]. As in Part I, we consider the case where the frequency scales at arbitrarily increasing rates relative to $n$. From a common empirical model of the absorption coefficient $a(f)$ for $f$ [11,12], it follows that

$$a(f) = \Theta\left(e^{c_1 f^2}\right) \tag{2}$$

for some constant $c_1 > 0$ independent of $n$.[1] The noise $n_i$ at node $i \in \{1, \cdots, n\}$ is assumed to be circularly symmetric complex additive colored Gaussian with zero mean and power spectral density (PSD) $N(f)$, and thus to be frequency-dependent. Since the PSD $N(f)$ decays linearly on the logarithmic scale in the operating frequency region [11, 13], its approximation is given by

$$N(f) = \Theta\left(\frac{1}{f^{a_0}}\right) \tag{3}$$

in terms of $f$ increasing with $n$, where $a_0 > 0$ is some constant independent of $n$. From (2) and (3), we may then have the following relationship between the absorption $a(f)$ and the noise PSD $N(f)$:

$$N(f) = \Theta\left(\frac{1}{(\log a(f))^{a_0/2}}\right). \tag{4}$$

From the narrow-band assumption, the received signal $y_k$ at node $k \in \{1, \cdots, n\}$ at a given time instance is given by

$$y_k = \sum_{i \in I} h_{ki} x_i + n_k,$$

where

$$h_{ki} = \frac{e^{j\theta_{ki}}}{\sqrt{A(r_{ki}, f)}} \tag{5}$$

represents the complex channel between nodes $i$ and $k$, $x_i \in \mathbb{C}$ is the signal transmitted by node $i$, and $I \subset \{1, \cdots, n\}$ is the set of simultaneously transmitting nodes. The random phases $\theta_{ki}$ are uniformly distributed over $[0, 2\pi)$ and independent for different $i$, $k$, and time. We thus assume a narrow-band time-varying channel.

Based on the above channel characteristics, operating regimes of the network are identified according to the following physical parameters: the absorption $a(f)$ and the noise PSD $N(f)$ which are exploited here by choosing the frequency $f$ based on the number of nodes, $n$. In other words, if the relationship between $f$ and $n$ is specified, then $a(f)$ and $N(f)$ can be given by a certain scaling function of $n$ from (2) and (3), respectively.

Further details on the system and channel models can be found in Section 2 of Part I.

## 3 Cut-set Upper Bound

To access the fundamental limits of a dense underwater network, a new cut-set upper bound on the total throughput scaling is analyzed from an information-theoretic perspective [14]. Consider a given cut $L$ dividing the network area

[1] We use the following notation: i) $f(x) = O(g(x))$ means that there exist constants $C$ and $c$ such that $f(x) \le Cg(x)$ for all $x > c$. ii) $f(x) = o(g(x))$ means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. iii) $f(x) = \Omega(g(x))$ if $g(x) = O(f(x))$. iv) $f(x) = \omega(g(x))$ if $g(x) = o(f(x))$. v) $f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $g(x) = O(f(x))$ [10].

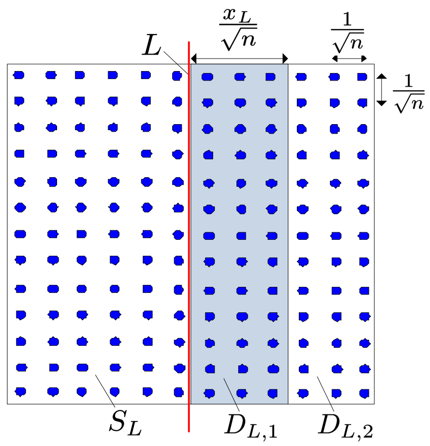


**Fig. 1** The cut $L$ in a two-dimensional dense regular network. $S_L$ and $D_L$ represent the sets of source and destination nodes, respectively, where $D_L$ is partitioned into two groups $D_{L,1}$ and $D_{L,2}$.

into two equal halves, as in [1,4,9] (see Fig. 1). Under the cut $L$, source nodes are on the left, while all nodes on the right are destinations. In this case, we have an $\Theta(n)\times\Theta(n)$ multiple-input multiple-output (MIMO) channel between the two sets of nodes separated by the cut.

Unlike the extended network case (refer to Section 3 of Part I), it is necessary to narrow down the class of S–D pairs according to their Euclidean distance to obtain a tight upper bound in a dense network. In this section, we derive a new upper bound based on hybrid approaches that consider either the sum of the capacities of the multiple-input single-output (MISO) channel between transmitters and each receiver or the amount of power transferred across the network according to operating regimes, similarly as in [9].

For the cut $L$, the total throughput $T(n)$ for sources on the left half is bounded by the capacity of the MIMO channel between $S_L$ and $D_L$, corresponding to the sets of sources and destinations, respectively, and thus is given by

$$T(n) \le \max_{\mathbf{Q}_L \ge 0} E\left[\log\det\left(\mathbf{I}_{n/2} + \frac{1}{N(f)}\mathbf{H}_L\mathbf{Q}_L\mathbf{H}_L^H\right)\right], \tag{6}$$

where $\mathbf{H}_L$ is the matrix with entries $[\mathbf{H}_L]_{ki} = h_{ki}$ for $i \in S_L, k \in D_L$, and $\mathbf{Q}_L \in \mathbb{C}^{\Theta(n)\times\Theta(n)}$ is the positive semi-definite input signal covariance matrix whose $k$-th diagonal element satisfies $[\mathbf{Q}_L]_{kk} \le P$ for $k \in S_L$. In the extended network framework [1], upper bounding the capacity by the total received signal-to-noise ratio (SNR) yields a tight bound due to poor power connections for all the operating regimes. In a dense network, however, we may have arbitrarily high received SNR for nodes in the set $D_L$ that are located close to the cut, or even for all the nodes, depending on the path-loss attenuation regimes, and thus need the separation between destination nodes that are well- and ill-connected to the left-half network in terms of power. More precisely, the set $D_L$ of destinations is partitioned into two groups $D_{L,1}$ and $D_{L,2}$ according to their location, as illustrated in Fig. 1. Then, by applying generalized Hadamard's inequality [15], we then have

$$\begin{aligned} T(n) &\le E\left[\log\det\left(\mathbf{I}_{n/2}+\frac{P}{N(f)}\mathbf{H}_L\mathbf{H}_L^H\right)\right] \\ &\le E\left[\log\det\left(\mathbf{I}_{|D_{L,1}|}+\frac{P}{N(f)}\mathbf{H}_{L,1}\mathbf{H}_{L,1}^H\right)\right] \\ &+E\left[\log\det\left(\mathbf{I}_{|D_{L,2}|}+\frac{P}{N(f)}\mathbf{H}_{L,2}\mathbf{H}_{L,2}^H\right)\right], \end{aligned} \tag{7}$$

where $\mathbf{H}_{L,l}$ is the matrix with entries $[\mathbf{H}_{L,l}]_{ki} = h_{ki}$ for $i \in S_L$, $k \in D_{L,l}$, and $l = 1, 2$. Here, the first inequality comes from the fact that Lemmas 1 and 2 of Part I also hold for a dense network, i.e., the covariance matrix $\mathbf{Q}_L$ maximizing the upper bound (6) is given by the diagonal $\tilde{\mathbf{Q}}_L$ with entries $[\tilde{\mathbf{Q}}_L]_{kk} = P$ for $k \in S_L$ (see [1] for more description). Note that the first and second terms in the right-hand side (RHS) of (7) represent the information transfer from $S_L$ to $D_{L,1}$ and from $S_L$ to $D_{L,2}$, respectively. Here, $D_{L,1}$ denotes the set of destinations located on the rectangular slab of width $x_L/\sqrt{n}$ immediately to the right of the centerline (cut), where $x_L \in \{0, 1, \cdots, \sqrt{n}/2\}$. The set $D_{L,2}$ is given by $D_L \setminus D_{L,1}$. It then follows that $|D_{L,1}| = x_L\sqrt{n}$ and $|D_{L,2}| = (\sqrt{n}/2 - x_L)\sqrt{n}$.

Let $T_l(n)$ denote the $l$-th term in the RHS of (7) for $l \in \{1, 2\}$. It is then reasonable to bound $T_1(n)$ by the cardinality of the set $D_{L,1}$ rather than the total received SNR. In contrast, using the power transfer argument for the term $T_2(n)$, as in the extended network case, will lead to a tight upper bound. It is thus important to set the parameter $x_L$ according to the attenuation parameter $a(f)$, based on the selection rule for $x_L$ [9], so that only $D_{L,1}$ contains the destination nodes with high received SNR. To be specific, we need to decide whether the SNR received by a destination $k \in D_L$ from the set $S_L$ of sources, denoted by $P_L^{(k)}/N(f)$, is larger than one, because degrees of freedom (also known as capacity pre-log factor) of the MISO channel are limited to one. If destination node $k$ has the total received SNR greater than one, i.e., $P_L^{(k)} = \omega(N(f))$, then it belongs to $D_{L,1}$. Otherwise, it follows that $k \in D_{L,2}$.

For analytical tractability, suppose that

$$a(f) = \Theta\left((1+\epsilon_0)^{n^\beta}\right) \text{ for } \beta \in [0, \infty), \tag{8}$$

where $\epsilon_0 > 0$ is an arbitrarily small constant, independent of $n$, which represents all the operating regimes with varying $\beta$. As in the extended network case, we define the following parameter

$$P_L^{(k)} = \frac{P}{c_0}\sum_{i\in S_L} r_{ki}^{-\alpha} a(f)^{-r_{ki}} \tag{9}$$

for some constant $c_0 > 0$ independent of $n$, which corresponds to the total power received from the signal sent by all the sources $i \in S_L$ at node $k$ on the right (see (1) and (5)). Let us index the node positions such that the source and destination nodes are located at positions $\left(\frac{-i_x+1}{\sqrt{n}}, \frac{i_y}{\sqrt{n}}\right)$ and $\left(\frac{k_x}{\sqrt{n}}, \frac{k_y}{\sqrt{n}}\right)$, respectively, for $i_x, k_x = 1, \cdots, \sqrt{n}/2$ and $i_y, k_y = 1, \cdots, \sqrt{n}$. We then obtain the following scaling results for $P_L^{(k)}$ as shown below.

**Lemma 1** *In a dense regular network, the term $P_L^{(k)}$ in (9) is upper- and lower-bounded by*

$$P_L^{(k)} = \begin{cases} O(n) & \text{if } 1 \le \alpha < 2 \text{ and } k_x = o\left(n^{1/2-\beta+\epsilon}\right) \\ O(n \log n) & \text{if } \alpha = 2 \text{ and } k_x = o\left(n^{1/2-\beta+\epsilon}\right) \\ O\left(\frac{n^{\alpha/2}}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}} \max\left\{1, n^{1/2-\beta}\right\}\right) & \text{if } k_x = \Omega\left(n^{1/2-\beta+\epsilon}\right) \end{cases} \tag{10}$$

*and*

$$P_L^{(k)} = \begin{cases} \Omega\left(\frac{n^{\alpha/2-\epsilon}}{k_x^{\alpha-1}}\right) & \text{if } k_x = o\left(n^{1/2-\beta+\epsilon}\right) \\ \Omega\left(\frac{1}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}} \max\left\{1, \frac{n^{1/2-\beta}}{(1+\epsilon_0)^{n^{\beta-1/2}}}\right\}\right) & \text{if } k_x = \Omega\left(n^{1/2-\beta+\epsilon}\right), \end{cases} \tag{11}$$

*respectively, for arbitrarily small positive constants $\epsilon$ and $\epsilon_0$, where $k_x/\sqrt{n}$ is the horizontal coordinate of node $k \in D_{L,2}$.*

The proof of this lemma is presented in Appendix A.1. Although the upper and lower bounds for $P_L^{(k)}$ are not identical to each other, showing these scaling results is sufficient to make a decision on $x_L$ according to the operating regimes. It is seen from Lemma 1 that when $k_x = o\left(n^{1/2-\beta+\epsilon}\right)$, $P_L^{(k)}$ does not depend on the parameter $\beta$ (or $a(f)$), while for $k_x = \Omega\left(n^{1/2-\beta+\epsilon}\right)$, node $k \in D_{L,2}$ gets ill-connected to the left half in terms of power since $P_L^{(k)}$ decreases exponentially with $n$. More specifically, when $k_x = o\left(n^{1/2-\beta+\epsilon}\right)$, it follows that $P_L^{(k)} = \omega(n^{\alpha\beta})$ from (11), resulting in $P_L^{(k)} = \omega(N(f))$ due to $N(f) = O(1)$. In contrast, under the condition $k_x = \Omega\left(n^{1/2-\beta+\epsilon}\right)$, it is observed from (10) that $P_L^{(k)}$ is exponentially decaying as a function of $n$, thus leading to $P_L^{(k)} = o(N(f))$. As a consequence, using the result of Lemma 1, three different regimes are identified and the selected $x_L$ is specified accordingly:

$$x_L = \begin{cases} \sqrt{n}/2 & \text{if } \beta = 0 \\ n^{1/2-\beta+\epsilon} & \text{if } 0 < \beta \le 1/2 \\ 0 & \text{if } \beta > 1/2 \end{cases} \tag{12}$$

for an arbitrarily small $\epsilon > 0$. It is now possible to show the proposed cut-set upper bound in dense networks.

**Theorem 1** *Consider an underwater regular network of unit area. Then, the upper bound on the total throughput $T(n)$ is given by*

$$T(n) = \begin{cases} O(n \log n) & \text{if } \beta = 0 \\ O\left(n^{1-\beta+\epsilon} \log n\right) & \text{if } 0 < \beta \leq 1/2 \\ O\left(\frac{n^{(1+\alpha+\beta a_0)/2}}{(1+\epsilon_0)^{n^{\beta-1/2}}}\right) & \text{if } \beta > 1/2, \end{cases} \tag{13}$$

*where $\epsilon$ and $\epsilon_0$ are arbitrarily small positive constants, and $a_0$ and $\beta$ are defined in (3) and (8), respectively.*

*Proof* We first compute the first term $T_1(n)$ in (7), focusing on the case for $0 \leq \beta \leq 1/2$ since otherwise $T_1(n) = 0$. Since the nodes in the set $D_{L,1}$ have good power connections to the left-half network and the information transfer to $D_{L,1}$ is limited in bandwidth (but not power), the term $T_1(n)$ is upper-bounded by the sum of the capacities of the MISO channels. More specifically, by generalized Hadamard's inequality [15], $T_1(n)$ can be easily bounded by

$$\begin{aligned} T_1(n) &\leq \sum_{k \in D_{L,1}} \log\left(1 + \frac{P}{N(f)} \sum_{i \in S_L} \frac{1}{A(r_{ki}, f)}\right) \\ &\leq x_L \sqrt{n} \log\left(1 + \frac{P n^{\alpha/2+1}}{a(f)^{1/\sqrt{n}} N(f)}\right) \\ &\leq x_L \sqrt{n} \log\left(1 + \frac{P n^{\alpha/2+1}}{N(f)}\right) \\ &\leq c_2 x_L \sqrt{n} \log n \end{aligned} \tag{14}$$

for some constant $c_2 > 0$ independent of $n$, where the last two steps are obtained from the fact that $0 < a(f) \leq 1$ and $N(f)$ tends to decrease polynomially with $n$ from the relation in (4). The upper bound for the second term $T_2(n)$ in (7) is now derived under the condition $\beta \in (0, \infty)$. From the fact that $\log(1 + x) \leq x$ for any $x$, it follows that

$$\begin{aligned} T_2(n) &\leq \sum_{k \in D_{L,2}} \log\left(1 + \frac{P}{N(f)} \sum_{i \in S_L} \frac{1}{A(r_{ki}, f)}\right) \\ &\leq \sum_{k \in D_{L,2}} \sum_{i \in S_L} \frac{P}{A(r_{ki}, f) N(f)} \\ &= \frac{1}{N(f)} \sum_{k \in D_{L,2}} P_L^{(k)}, \end{aligned} \tag{15}$$

which corresponds to the sum of the total received SNR from the left-half network to the destination set $D_{L,2}$. Hence, combining the two bounds (14) and (15) along with the choices for $x_L$ specified in (12), we obtain the following upper bound on the total throughput $T(n)$:

$$T(n) \le \begin{cases} c_2 n \log n & \text{if } \beta = 0 \\ c_2 n^{1-\beta+\epsilon} \log n + \frac{1}{N(f)} \sum\limits_{k \in D_{L,2}} P_L^{(k)} & \text{if } 0 < \beta \le 1/2 \\ \frac{1}{N(f)} \sum\limits_{k \in D_L} P_L^{(k)} & \text{if } \beta > 1/2 \end{cases}$$
$$= \begin{cases} c_2 n \log n & \text{if } \beta = 0 \\ c_2 n^{1-\beta+\epsilon} \log n + \frac{1}{N(f)} \sum_{k_x = x_L}^{\sqrt{n}/2} \sum_{k_y=1}^{\sqrt{n}} \frac{n^{\alpha/2+1/2-\beta}}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}} & \text{if } 0 < \beta \le 1/2 \\ \frac{1}{N(f)} \sum_{k_x=1}^{\sqrt{n}/2} \sum_{k_y=1}^{\sqrt{n}} \frac{n^{\alpha/2}}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}} & \text{if } \beta > 1/2 \end{cases}$$
$$\le \begin{cases} c_2 n \log n & \text{if } \beta = 0 \\ c_3 n^{1-\beta+\epsilon} \log n & \text{if } 0 < \beta \le 1/2 \\ \frac{n^{(1+\alpha)/2}}{N(f)} \sum_{k_x=1}^{\sqrt{n}/2} \frac{1}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}} & \text{if } \beta > 1/2 \end{cases} \quad (16)$$

for an arbitrarily small $\epsilon > 0$ and some constant $c_3 > 0$ independent of $n$, where the equality comes from (10). The second inequality holds due to the fact that the term $\frac{n^{\alpha/2}}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}}$ tends to decay exponentially with $n$ under the conditions $0 < \beta \le 1/2$ and $x_L \le k_x \le \sqrt{n}/2$, and thus the total is dominated by the information transfer $T_1(n)$ in (14). Now let us focus on the last line of (16), which corresponds to the total amount of SNR received by all nodes for the condition $\beta > 1/2$. For $\beta > 1/2$, using the two relationships (4) and (8) follows that

$$\frac{n^{(1+\alpha)/2}}{N(f)} \sum_{k_x=1}^{\sqrt{n}/2} \frac{1}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}} \le \frac{n^{(1+\alpha)/2}}{N(f)} \frac{1}{(1+\epsilon_0)^{n^{\beta-1/2}} - 1}$$
$$\le \frac{c_4 n^{(1+\alpha+\beta a_0)/2}}{(1+\epsilon_0)^{n^{\beta-1/2}}}$$

for some constant $c_4 > 0$ independent of $n$, where the second inequality holds due to $(1+\epsilon_0)^{n^{\beta-1/2}} = \omega(1)$ under the condition. This coincides with the result shown in (13), which completes the proof.

Note that the upper bound [4] for wireless radio networks of unit area is given by $O(n \log n)$, which is the same as the case with $\beta = 0$ (or equivalently $a(f) = \Theta(1)$) in the dense underwater network. Now let us discuss the fundamental limits of the network according to three different operating regimes shown in (13).

*Remark 1* The upper bound on the total capacity scaling is illustrated in Fig. 2 versus the parameter $\beta$ (logarithmic terms are omitted for convenience). We first address the regime $\beta = 0$ (i.e., low path-loss attenuation regime), in which the upper bound on $T(n)$ is active with $x_L = \sqrt{n}/2$, or equivalently $D_{L,1} = D_L$, while $T_2(n) = 0$. In this case, the total throughput of the network is limited by the degrees of freedom of the $\Theta(n) \times \Theta(n)$ MIMO channel between $S_L$ and

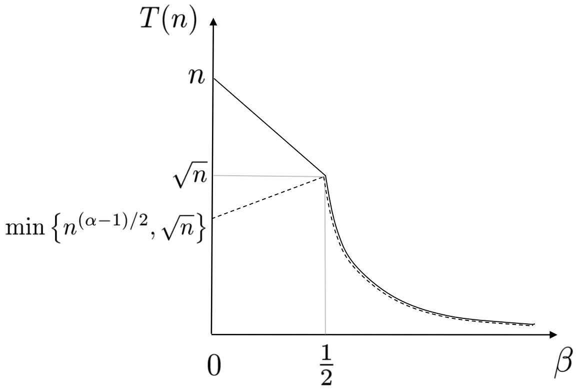


**Fig. 2** Upper (solid) and lower (dashed) bounds on the capacity scaling $T(n)$.

$D_L$, and is roughly linear in the bandwidth, thus resulting in a bandwidth-limited network. In particular, our interest is in the operating regimes for which the network becomes power-limited as $\beta > 0$. In the second regime $0 < \beta \le 1/2$ (i.e., medium path-loss attenuation regime), as pointed out in the proof of Theorem 1, the upper bound on $T(n)$ is dominated by the information transfer from $S_L$ to $D_{L,1}$, that is, the term $T_1(n)$ in (14) contributes more than $T_2(n)$ in (15). The total throughput is thus limited by the degrees of freedom of the MIMO channel between $S_L$ and $D_{L,1}$, since more available bandwidth leads to an increment in $T_1(n)$. As a consequence, in this regime, the network is both bandwidth- and power-limited. In the third regime $\beta > 1/2$ (i.e., high path-loss attenuation regime), the upper bound (15) is active with $x_L = 0$, or equivalently $D_{L,2} = D_L$, while $T_1(n) = 0$. The information transfer to $D_L$ is thus totally limited by the sum of the total received SNR from the left-half network to the destination nodes in $D_L$. In other words, in the third regime, the network is limited in power, but not in bandwidth.

Note that the upper bound on $T(n)$ decays polynomially with increasing $\beta$ in the regime $0 < \beta \le 1/2$, while it drops off exponentially when $\beta > 1/2$. In addition, two other expressions on the total throughput $T(n)$ are summarized as follows.

*Remark 2* From (4) and (8), the upper bound and the corresponding operating regimes can also be presented below in terms of the attenuation parameter $a(f)$:

$$T(n) = \begin{cases} O(n \log n) & \text{if } a(f) = \Theta(1) \\ O\left(\frac{n^{1+\epsilon} \log n}{\log a(f)}\right) & \text{if } a(f) = \omega(1) \text{ and } a(f) = O\left((1+\epsilon_0)^{\sqrt{n}}\right) \\ O\left(\frac{n^{(1+\alpha)/2} (\log a(f))^{a_0/2}}{a(f)^{1/\sqrt{n}}}\right) & \text{if } a(f) = \omega\left((1+\epsilon_0)^{\sqrt{n}}\right). \end{cases}$$

Note that as $a(f) = \omega\left((1+\epsilon_0)^{\sqrt{n}}\right)$, we also obtain

$$T(n) = O\left(\frac{n^{(1+\alpha)/2}}{a(f)^{1/\sqrt{n}}N(f)}\right),$$

which is expressed as a function of the spreading factor $\alpha$ as well as the absorption $a(f)$ and the noise PSD $N(f)$. Using (2) and (3) further yields the following expression

$$T(n) = \begin{cases} O(n\log n) & \text{if } f = \Theta(1) \\ O\left(\frac{n^{1+\epsilon}\log n}{f^2}\right) & \text{if } f = \omega(1) \text{ and } f = O\left(n^{1/4}\right) \\ O\left(\frac{n^{(1+\alpha)/2}f^{a_0}}{e^{c_1 f^2/\sqrt{n}}}\right) & \text{if } f = \omega\left(n^{1/4}\right), \end{cases}$$

which represents the upper bound for the operating regimes identified by frequency scaling.

## 4 Achievability Result

In this section, to show the order optimality of underwater networks, we analyze the achievable throughput scaling for dense networks. Based on the earlier studies [4, 7, 9, 16] for wireless radio networks, it follows that using MH routing [7] is preferred at power-limited regimes, while a hierarchical cooperation (HC) strategy [4] may have a better performance at bandwidth-limited regimes. Existing transmission schemes thus need to be used carefully, depending on path-loss attenuation regimes. However, utilizing long-distance transmissions (e.g., the schemes associated with HC [4, 9]) will not be suitable for underwater networks since it can cause extremely long delays in receiving the desired data.[2] For this reason, we mainly focus on the achievability based on the nearest-neighbor MH routing. In a dense regular network, we identify the operating regimes for which the achievable throughput matches the upper bound shown in Section 3.

As in the extended network case (refer to Section 4 of Part I), the nearest-neighbor MH routing [7] is used with a slight modification. The basic procedure of the MH protocol under our dense regular network is briefly described as follows:

- Divide the network into $n$ square routing cells, each of which has the same area.
- Draw a line connecting an S–D pair.
- At each node, use the transmit power of

$$P\min\left\{1, \frac{a(f)^{1/\sqrt{n}}N(f)}{n^{\alpha/2}}\right\}.$$

[2] Note that in acoustic communication, the transmission delay is usually much higher than the processing delay at each node. The propagation speed of acoustic signal in water is around 1500 m/s [17], which is five orders of magnitude lower than that of radio wave.

The scheme operates with the full power when $a(f) = \Omega\left(\frac{n^{\alpha\sqrt{n}/2}}{N(f)^{\sqrt{n}}}\right)$. On the other hand, when $a(f) = o\left(\frac{n^{\alpha\sqrt{n}/2}}{N(f)^{\sqrt{n}}}\right)$, the transmit power $Pa(f)^{1/\sqrt{n}}N(f)/n^{\alpha/2}$, which scales slower than $\Theta(1)$, is sufficient so that the received SNR at each node is bounded by 1 (note that having a higher power is unnecessary in terms of throughput improvement).

The amount of interference is now quantified to show the achievable throughput based on MH.

**Lemma 2** *Consider a dense regular network that uses the nearest-neighbor MH protocol. Then, the total interference power $P_I$ from other simultaneously transmitting nodes, corresponding to the set $I \subset \{1, \cdots, n\}$, is bounded by*

$$P_I = \begin{cases} O\left(\frac{\max\left\{n^{(1/2-\beta)(2-\alpha)}, \log n\right\}}{n^{\beta a_0/2}}\right) & \text{if } 0 \le \beta < 1/2 \\ O\left(n^{-\beta a_0/2}\right) & \text{if } \beta = 1/2 \\ O\left(\frac{n^{\alpha/2}}{(1+\epsilon_0)^{n^{\beta-1/2}}}\right) & \text{if } \beta > 1/2 \end{cases} \tag{17}$$

*for an arbitrarily small $\epsilon_0 > 0$, where $a_0$ and $\beta$ are defined in (3) and (8), respectively.*

The proof of this lemma is presented in Appendix A.2. From (4) and (8), we note that when $\beta = 1/2$, it follows that $P_I = O(N(f))$, i.e., $P_I$ is upper-bounded by the PSD $N(f)$ of noise. Using Lemma 2, a lower bound on the capacity scaling can be derived, and hence the following result shows the achievable rates under the MH protocol in a dense network.

**Theorem 2** *In an underwater regular network of unit area,*

$$T(n) = \begin{cases} \Omega\left(\frac{\sqrt{n}}{\max\left\{n^{(1/2-\beta)(2-\alpha)}, \log n\right\}}\right) & \text{if } 0 \le \beta < 1/2 \\ \Omega\left(\sqrt{n}\right) & \text{if } \beta = 1/2 \\ \Omega\left(\frac{n^{(1+\alpha+\beta a_0)/2}}{(1+\epsilon_0)^{n^{\beta-1/2}}}\right) & \text{if } \beta > 1/2 \end{cases} \tag{18}$$

*is achievable.*

*Proof* Suppose that the nearest-neighbor MH protocol described above is used. Then, from (1), the received signal power $P_r$ from the desired transmitter is given by

$$P_r = \frac{P \min\left\{1, \frac{a(f)^{1/\sqrt{n}}N(f)}{n^{\alpha/2}}\right\} n^{\alpha/2}}{c_0 a(f)^{1/\sqrt{n}}},$$

which can be rewritten as

$$\begin{aligned} & \frac{Pn^{\alpha/2}}{c_0(1+\epsilon_0)^{n^{\beta-1/2}}} \min\left\{1, \frac{(1+\epsilon_0)^{n^{\beta-1/2}}}{n^{(\alpha+\beta a_0)/2}}\right\} \\ &= \begin{cases} \frac{P}{c_0 n^{\beta a_0/2}} & \text{if } 0 \le \beta \le 1/2 \\ \frac{Pn^{\alpha/2}}{c_0(1+\epsilon_0)^{n^{\beta-1/2}}} & \text{if } \beta > 1/2 \end{cases} \end{aligned} \tag{19}$$

with respect to the parameter $\beta$ using (4) and (8). To obtain a lower bound on the capacity scaling, the signal-to-interference-and-noise ratio (SINR) seen by receiver $i \in \{1, \cdots, n\}$ can be computed using (17) and (19). By assuming the worst case noise, which lower-bounds the transmission rate, and full CSI at the receiver, the achievable throughput per S–D pair is then lower-bounded by

$$
\begin{aligned}
&\log(1+\mathrm{SINR}) \\
&= \log\left(1+\frac{P_r}{N(f)+P_I}\right) \\
&= \begin{cases} \Omega\left(\log\left(1+\frac{1}{\max\left\{n^{(1/2-\beta)(2-\alpha)},\log n\right\}}\right)\right) & \text{if } 0 \le \beta < 1/2 \\ \Omega(1) & \text{if } \beta = 1/2 \\ \Omega\left(\log\left(1+\frac{n^{(\alpha+\beta a_0)/2}}{(1+\epsilon_0)^{n^{\beta-1/2}}}\right)\right) & \text{if } \beta > 1/2 \end{cases} \\
&= \begin{cases} \Omega\left(\frac{1}{\max\left\{n^{(1/2-\beta)(2-\alpha)},\log n\right\}}\right) & \text{if } 0 \le \beta < 1/2 \\ \Omega(1) & \text{if } \beta = 1/2 \\ \Omega\left(\frac{n^{(\alpha+\beta a_0)/2}}{(1+\epsilon_0)^{n^{\beta-1/2}}}\right) & \text{if } \beta > 1/2, \end{cases}
\end{aligned}
$$

where the second equality holds since $N(f) = \Theta(n^{-\beta a_0/2})$ and thus $P_I = \omega(N(f))$ for $0 \le \beta < 1/2$, $P_I = \Theta(P_r) = \Theta(N(f))$ for $\beta = 1/2$, and $P_I = o(N(f))$ for $\beta > 1/2$. The last equality comes from the fact that $\log(1+x) = (\log e)x + O(x^2)$ for small $x > 0$. Since there are $\Omega(\sqrt{n})$ S–D pairs that can be active simultaneously in the network, the total throughput is finally given by (18), which completes the proof.

Note that the achievable throughput [7] for wireless radio networks of unit area using MH routing is given by $\Omega(\sqrt{n})$, which is the same as the case for which $\beta = 1/2$ (or equivalently $a(f) = \Theta\left((1+\epsilon_0)^{\sqrt{n}}\right)$) in a dense underwater network. The lower bound on the total throughput $T(n)$ is also shown in Fig. 2 according to the parameter $\beta$. From this result, an interesting observation follows. To be specific, in the regime $0 \le \beta \le 1/2$, the lower bound on $T(n)$ grows linearly with increasing $\beta$, because the total interference power $P_I$ in (17) tends to decrease as $\beta$ increases. In this regime, note that $P_I = \Omega(P_r)$. Meanwhile, when $\beta > 1/2$, the lower bound reduces rapidly due to the exponential path-loss attenuation in terms of increasing $\beta$.

In addition, similarly as in Section 3, two other expressions on the achievability result are now summarized as in the following.

*Remark 3* From (4) and (8), the lower bound on the throughput $T(n)$ and the corresponding operating regimes can also be presented below in terms of the attenuation parameter $a(f)$:

$$T(n) = \begin{cases} \Omega\left(\frac{\sqrt{n}}{\max\left\{a(f)^{(2-\alpha)/\sqrt{n}}, \log n\right\}}\right) & \text{if } a(f) = \Omega(1) \text{ and } a(f) = o\left((1+\epsilon_0)^{\sqrt{n}}\right) \\ \Omega\left(\sqrt{n}\right) & \text{if } a(f) = \Theta\left((1+\epsilon_0)^{\sqrt{n}}\right) \\ \Omega\left(\frac{n^{(1+\alpha)/2}(\log a(f))^{a_0/2}}{a(f)^{1/\sqrt{n}}}\right) & \text{if } a(f) = \omega\left((1+\epsilon_0)^{\sqrt{n}}\right). \end{cases}$$

Furthermore, using (2) and (3) follows that

$$T(n) = \begin{cases} \Omega\left(\frac{\sqrt{n}}{\max\left\{e^{c_1(2-\alpha)f^2/\sqrt{n}}, \log n\right\}}\right) & \text{if } f = \Omega(1) \text{ and } f = o\left(n^{1/4}\right) \\ \Omega\left(\sqrt{n}\right) & \text{if } f = \Theta\left(n^{1/4}\right) \\ \Omega\left(\frac{n^{(1+\alpha)/2} f^{a_0}}{e^{c_1 f^2/\sqrt{n}}}\right) & \text{if } f = \omega\left(n^{1/4}\right), \end{cases}$$

which represents the lower bound for the operating regimes obtained from the relationship between the frequency $f$ and the number $n$ of nodes.

Now let us turn to examining how the upper bound shown in Section 3 is close to the achievable throughput scaling.

*Remark 4* Based on Theorems 1 and 2, it is seen that if $\beta \geq 1/2$, then the achievable rate of the MH protocol is close to the upper bound up to $n^\epsilon$ for an arbitrarily small $\epsilon > 0$ (note that the two bounds are of exactly the same order especially for $\beta > 1/2$). The condition $\beta \geq 1/2$ corresponds to the high path-loss attenuation regime, and is equivalent to $a(f) = \Omega\left((1+\epsilon_0)^{\sqrt{n}}\right)$ or $f = \Omega\left(n^{1/4}\right)$. Therefore, the MH is order-optimal in regular networks of unit area under the aforementioned operating regimes, whereas in extended networks, using MH routing results in the order optimality for all the operating regimes.

Finally, we remark that applying the HC strategy [4] does not guarantee the order optimality in the regime $0 \leq \beta < 1/2$ (i.e., low and medium path-loss attenuation regimes). The primary reason is specified under each operating regime: for the condition $\beta = 0$, following the steps similar to those of Lemma 2, it follows that $P_I = \omega(P_r)$ at all levels of the hierarchy, thereby resulting in SINR $= o(1)$ for each transmission (the details are not shown in this paper). It is thus not possible to achieve a linear throughput scaling. Now let us focus on the case where $0 < \beta < 1/2$. At the top level of the hierarchy, the transmissions between two clusters having distance $O(1)$ becomes a bottleneck because of the exponential path-loss attenuation with propagation distance. Hence, the achievable throughput of the HC decays exponentially with respect to $n$, which is significantly lower than that in (18).

## 5 Conclusion

In Part II in a two-part series, dense underwater acoustic networks were analyzed in terms of capacity scaling. In dense networks, the upper bound was derived characterizing three different operating regimes, in which there exists either a bandwidth or power limitation, or both. As in Part I, the nearest-neighbor MH protocol was introduced with a slight modification to show the achievability result—the MH protocol is order-optimal in power-limited regimes (i.e., the case where the frequency $f$ scales faster than or as $n^{1/4}$) of dense networks. Therefore, it turned out that there exists a right frequency scaling that makes our scaling results for underwater acoustic networks to break free from scaling limitations related to the channel characteristics that were described in [18]. For all the operating regimes, the exact capacity scaling of dense underwater networks remains still open, whereas the order optimality of extended underwater networks has been proved in Part I.

For a more in-depth discussion of the extended network results, see the conclusion section of Part I.

## A Appendix

### A.1 Proof of Lemma 1

Upper and lower bounds on $P_L^{(k)}$ in a dense network are derived by basically following the same node indexing and layering techniques as those in Part I. We refer to Appendix A.2 and Fig. 4 in [1] for the detailed description (note that the destination nodes are, however, located at positions $\left(\frac{k_x}{\sqrt{n}}, \frac{k_y}{\sqrt{n}}\right)$ in dense networks). Similarly to the extended network case, from (9), the term $P_L^{(k)}$ is then given by

$$P_L^{(k)} = \frac{P}{c_0} \sum_{i_x=1}^{\sqrt{n}/2} \sum_{i_y=1}^{\sqrt{n}} \frac{n^{\alpha/2}}{((i_x + k_x - 1)^2 + (i_y - k_y)^2)^{\alpha/2} \, a(f)^{\sqrt{((i_x+k_x-1)^2+(i_y-k_y)^2)/n}}}.$$

First, focus on how to obtain an upper bound for $P_L^{(k)}$. Assuming that all the nodes in each layer are moved onto the innermost boundary of the corresponding ring, we then have

$$\begin{aligned} P_L^{(k)} &\le \frac{Pn^{\alpha/2}}{c_0} \sum_{i'=k_x}^{k_x+\sqrt{n}/2-1} \frac{c_5(i'+1)}{i'^{\alpha} a(f)^{i'\sqrt{n}}} \\ &\le \frac{2c_5 Pn^{\alpha/2}}{c_0} \sum_{i'=k_x}^{\sqrt{n}} \frac{1}{i'^{\alpha-1} a(f)^{i'\sqrt{n}}} \\ &= c_6 Pn^{\alpha/2} \sum_{i'=k_x}^{\sqrt{n}} \frac{1}{i'^{\alpha-1} \left(1+\epsilon_0\right)^{i' n^{\beta-1/2}}} \end{aligned} \quad (20)$$

for some positive constants $c_0$, $c_5$, and $c_6$ independent of $n$ and an arbitrarily small $\epsilon_0 > 0$, where the equality comes from the relationship (8) between $a(f)$ and $\beta$. We first consider the case where $k_x = o\left(n^{1/2-\beta+\epsilon}\right)$ for an arbitrarily small $\epsilon > 0$. Under this condition, from the fact that the term $i'^{\alpha-1}$ in the RHS of (20) is dominant in terms of upper-bounding $P_L^{(k)}$ for $i' = k_x, \cdots, \sqrt{n}$, (20) is further bounded by

$$\begin{aligned} P_L^{(k)} &\le c_6 P n^{\alpha/2} \sum_{i'=k_x}^{\sqrt{n}} \frac{1}{i'^{\alpha-1}} \\ &\le c_6 P n^{\alpha/2} \left( \frac{1}{k_x^{\alpha-1}} + \int_{k_x}^{\sqrt{n}} \frac{1}{x^{\alpha-1}} dx \right), \end{aligned}$$

which yields $P_L^{(k)} = O\left(n^{\alpha/2}(\sqrt{n})^{2-\alpha}\right) = O(n)$ for $1 \le \alpha < 2$ and $P_L^{(k)} = O(n \log n)$ for $\alpha = 2$. When $k_x = \Omega(n^{1/2-\beta+\epsilon})$, the upper bound (20) for $P_L^{(k)}$ is dominated by the term $(1+\epsilon_0)^{i' n^{\beta-1/2}}$, and thus is given by

$$\begin{aligned} P_L^{(k)} &\le c_6 P n^{\alpha/2} \sum_{i'=k_x}^{\sqrt{n}} \frac{1}{(1+\epsilon_0)^{i' n^{\beta-1/2}}} \\ &\le c_6 P n^{\alpha/2} \left( \frac{1}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}} + \int_{k_x}^{\sqrt{n}} \frac{1}{(1+\epsilon_0)^{x n^{\beta-1/2}}} dx \right) \\ &= c_6 P n^{\alpha/2} \left( \frac{1}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}} + \int_{1}^{\sqrt{n}/k_x} \frac{k_x}{(1+\epsilon_0)^{x k_x n^{\beta-1/2}}} dx \right) \\ &\le c_6 P n^{\alpha/2} \left( \frac{1}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}} + \frac{n^{1/2-\beta}}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}} \right) \\ &\le \frac{c_7 P n^{\alpha/2}}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}} \max\left\{1, n^{1/2-\beta}\right\} \end{aligned} \tag{21}$$

for some constant $c_7 > 0$ independent of $n$, which is the last result in (10).

Next, let us turn to deriving a lower bound for $P_L^{(k)}$. Since each layer has at least one node that is onto the innermost boundary of the corresponding ring, the lower bound similarly follows

$$\begin{aligned} P_L^{(k)} &\ge \frac{P n^{\alpha/2}}{c_0} \sum_{i'=k_x}^{k_x+\sqrt{n}/2-1} \frac{1}{i'^{\alpha} a(f)^{i'\sqrt{n}}} \\ &= c_6 P n^{\alpha/2} \sum_{i'=k_x}^{k_x+\sqrt{n}/2-1} \frac{1}{i'^{\alpha} (1+\epsilon_0)^{i' n^{\beta-1/2}}}. \end{aligned} \tag{22}$$

For the condition $k_x = o(n^{1/2-\beta+\epsilon})$, (22) is represented as

$$\begin{aligned} P_L^{(k)} &\ge c_6 P n^{\alpha/2} \sum_{i'=k_x}^{k_x+\sqrt{n}/2-1} \frac{1}{i'^{\alpha} (1+\epsilon_0)^{i' n^{\beta-1/2}}} \\ &\ge c_6 P n^{\alpha/2} \sum_{i'=k_x}^{2k_x-1} \frac{1}{i'^{\alpha} (1+\epsilon_0)^{i' n^{\beta-1/2}}} \\ &\ge \frac{c_6 P n^{\alpha/2}}{n^{\epsilon'}} \sum_{i'=k_x}^{2k_x-1} \frac{1}{i'^{\alpha}} \\ &\ge \frac{c_8 P n^{\alpha/2}}{n^{\epsilon'} k_x^{\alpha-1}} \end{aligned}$$

for an arbitrarily small $\epsilon' > 0$ and some constant $c_8 > 0$ independent of $n$, where the third inequality holds due to $(1+\epsilon_0)^{k_x n^{\beta-1/2}} = O(n^{\epsilon'})$. On the other hand, similarly as in the steps of (21), the condition $k_x = \Omega(n^{1/2-\beta+\epsilon})$ yields the following lower bound for $P_L^{(k)}$:

$$\begin{aligned}
P_L^{(k)} &\geq c_6 P \sum_{i'=k_x}^{k_x+\sqrt{n}/2-1} \frac{1}{(1+\epsilon_0)^{i' n^{\beta-1/2}}} \\
&\geq c_6 P \left( \frac{1}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}} + \int_{k_x+1}^{k_x+\sqrt{n}/2-1} \frac{1}{(1+\epsilon_0)^{x n^{\beta-1/2}}} dx \right) \\
&\geq c_9 P \left( \frac{1}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}} + \frac{n^{1/2-\beta}}{(1+\epsilon_0)^{(k_x+1) n^{\beta-1/2}}} \right) \\
&\geq \frac{c_9 P}{(1+\epsilon_0)^{k_x n^{\beta-1/2}}} \max \left\{ 1, \frac{n^{1/2-\beta}}{(1+\epsilon_0)^{n^{\beta-1/2}}} \right\}
\end{aligned}$$

some constant $c_9 > 0$ independent of $n$, which finally complete the proof of the lemma.

## A.2 Proof of Lemma 2

The layering technique used in Part I is similarly applied (see Fig. 2 in [1]). From (1), the total interference power $P_I$ at each node from simultaneously transmitting nodes is upper-bounded by

$$\begin{aligned}
P_I &= \sum_{k=1}^{\sqrt{n}} (8k) \frac{P \min\left\{1, \frac{a(f)^{1/\sqrt{n}} N(f)}{n^{\alpha/2}}\right\}}{c_0 (k/\sqrt{n})^{\alpha} a(f)^{k/\sqrt{n}}} \\
&= \frac{8 P n^{\alpha/2} \min\left\{1, \frac{a(f)^{1/\sqrt{n}} N(f)}{n^{\alpha/2}}\right\}}{c_0} \sum_{k=1}^{\sqrt{n}} \frac{1}{k^{\alpha-1} a(f)^{k/\sqrt{n}}}. \qquad (23)
\end{aligned}$$

Using (4) and (8), the upper bound (23) on $P_I$ can be expressed as

$$\begin{aligned}
P_I &\leq c_{10} P n^{\alpha/2} \min\left\{1, \frac{(1+\epsilon_0)^{n^{\beta-1/2}}}{n^{(\alpha+\beta a_0)/2}}\right\} \sum_{k=1}^{\sqrt{n}} \frac{1}{k^{\alpha-1}(1+\epsilon_0)^{k n^{\beta-1/2}}} \\
&\leq \begin{cases} \frac{c_{10} P}{n^{\beta a_0/2}} \sum_{k=1}^{\sqrt{n}} \frac{1}{k^{\alpha-1}(1+\epsilon_0)^{k n^{\beta-1/2}}} & \text{if } 0 \leq \beta < 1/2 \\ \frac{c_{10} P}{n^{\beta a_0/2}} \sum_{k=1}^{\sqrt{n}} \frac{1}{(1+\epsilon_0)^{k}} & \text{if } \beta = 1/2 \\ c_{10} P n^{\alpha/2} \sum_{k=1}^{\sqrt{n}} \frac{1}{(1+\epsilon_0)^{k n^{\beta-1/2}}} & \text{if } \beta > 1/2 \end{cases} \\
&\leq \begin{cases} \frac{c_{10} P}{n^{\beta a_0/2}} \sum_{k=1}^{\sqrt{n}} \frac{1}{k^{\alpha-1}(1+\epsilon_0)^{k n^{\beta-1/2}}} & \text{if } 0 \leq \beta < 1/2 \\ \frac{c_{10} P}{n^{\beta a_0/2}} \frac{1}{(1+\epsilon_0)-1} & \text{if } \beta = 1/2 \\ c_{10} P n^{\alpha/2} \frac{1}{(1+\epsilon_0)^{n^{\beta-1/2}}-1} & \text{if } \beta > 1/2 \end{cases} \\
&\leq \begin{cases} \frac{c_{10} P}{n^{\beta a_0/2}} \sum_{k=1}^{\sqrt{n}} \frac{1}{k^{\alpha-1}(1+\epsilon_0)^{k n^{\beta-1/2}}} & \text{if } 0 \leq \beta < 1/2 \\ \frac{c_{11} P}{n^{\beta a_0/2}} & \text{if } \beta = 1/2 \\ \frac{c_{11} P n^{\alpha/2}}{(1+\epsilon_0)^{n^{\beta-1/2}}} & \text{if } \beta > 1/2 \end{cases}
\end{aligned}$$

for some positive constants $c_{10}$ and $c_{11}$ independent of $n$. Based on the argument in Appendix A.1, when $0 \le \beta < 1/2$, it follows that

$$\begin{aligned}
\sum_{k=1}^{\sqrt{n}} \frac{1}{k^{\alpha-1}(1+\epsilon_0)^{kn^{\beta-1/2}}} &= \sum_{k=1}^{n^{1/2-\beta}-1} \frac{1}{k^{\alpha-1}(1+\epsilon_0)^{kn^{\beta-1/2}}} + \sum_{k=n^{1/2-\beta}}^{\sqrt{n}} \frac{1}{k^{\alpha-1}(1+\epsilon_0)^{kn^{\beta-1/2}}} \\
&\le \sum_{k=1}^{n^{1/2-\beta}-1} \frac{1}{k^{\alpha-1}} + \frac{1}{n^{(1/2-\beta)(\alpha-1)}} \sum_{k=n^{1/2-\beta}}^{\sqrt{n}} \frac{1}{(1+\epsilon_0)^{kn^{\beta-1/2}}} \\
&\le \left(1 + \int_1^{n^{1/2-\beta}} \frac{1}{x^{\alpha-1}} dx\right) \\
&\quad + \frac{1}{n^{(1/2-\beta)(\alpha-1)}} \left(\frac{1}{1+\epsilon_0} + \int_{n^{1/2-\beta}}^{\sqrt{n}} \frac{1}{(1+\epsilon_0)^{xn^{\beta-1/2}}} dx\right) \\
&\le 2\int_1^{n^{1/2-\beta}} \frac{1}{x^{\alpha-1}} dx + \frac{2}{n^{(1/2-\beta)(\alpha-1)}} \int_1^{n^{\beta}} \frac{n^{1/2-\beta}}{(1+\epsilon_0)^x} dx \\
&\le \begin{cases} 4n^{(1/2-\beta)(2-\alpha)} & \text{if } 1 \le \alpha < 2 \\ \log n & \text{if } \alpha = 2, \end{cases}
\end{aligned}$$

which results in $P_I = O\left(\frac{\max\left\{n^{(1/2-\beta)(2-\alpha)}, \log n\right\}}{n^{\beta a_0/2}}\right)$. This completes the proof of the lemma.

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